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GENERALIZATION OF TRACES IN PIVOTAL CATEGORIES

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ABSTRACT

It is known that the trace is a characteristic of the endomorphisms of a vector space, and it is an invariant of matrices representation of these endomorphisms. This notion was generalized in the context of a category, and its properties are used to construct some quantum topological invariants. In this article we show the generalization of the notion of a trace in the categories and some of its applications.

Keywords: *Trace, ribbon Hopf algebra, invariant of links.*

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1 Introduction

One can say that each model mathematics represents some facts or some structures which are from the domains of natural sciences, of society sciences and even from the variety of life. Working on the models allows one to find the knowledge which is helpful to improving the models and applying to industry, education, ... One of the most popular algebraic structures is a vector space over a field. That is a model containing the configurations called the vectors and the actions among the vectors. The vector spaces are the ground to define the algebraic structures more complicated, for example, the rings and the algebras. Let V be a finite dimensional vector space. The information about V can be described through its basis, its duality or the linear maps on itself. We pay attention to a characteristic of its linear maps called *trace*. The trace of an endomorphism f in $\mathrm{End}(V)$ can be determined by using matrix representation of f and it does not depend on the choice of the basis (see e.g., [5]). Other way to determine the trace is using some special structures of the category $\mathrm{Vect}_{\Bbbk}$ of finite dimensional vector spaces as in Theorem 2.4. This approach allows one to generalize the notion of a trace of an endomorphism in the context of categories (see [5, 3]). One found the interesting applications of trace, in particular in the construction of quantum invariants (see [7, 9]). In this article, we have an interest in systematizing the notion of a trace and its properties.

The text is organized in four sections. In Section 2, we recall some definitions and results about the trace in the category $\mathrm{Vect}_{\Bbbk}$ of finite dimensional vector spaces over a field $\Bbbk$. Section 3 shows the generalization and systematizes the notion of a trace in the context of the pivotal categories. Finally, in Section 4, we represent an application of the trace in the construction of quantum invariants.

2 Trace in category $\mathrm{Vect}_{\Bbbk}$

In this section we summarize some results about the trace and its properties in the category of finite dimensional vector spaces $\mathrm{Vect}_{\Bbbk}$.

Definition 2.1. *A category $\mathscr{C}$ is the following data:*

1) *a class of objects $Ob(\mathscr{C})$,*
2) *for every objects $X, Y \in Ob(\mathscr{C})$, the class $\mathrm{Hom}_{\mathscr{C}}(X, Y)$ of morphisms from X to Y,*
3) *for any objects $X, Y, Z \in Ob(\mathscr{C})$, a composition map $\mathrm{Hom}_{\mathscr{C}}(Y, Z) \times \mathrm{Hom}_{\mathscr{C}}(X, Y) \to \mathrm{Hom}_{\mathscr{C}}(X, Z)$, $(f, g) \mapsto f \circ g$ which satisfy the following axioms:*

a. The composition is associative, i.e., $(f \circ g) \circ h = f \circ (g \circ h)$,

b. For each $X \in Ob(\mathscr{C})$, *there is a morphism* $1_X \in \mathrm{Hom}_{\mathscr{C}}(X, X)$ *such that* $1_X \circ f = f$ *and* $g \circ 1_X = g$ *for any* $f \in \mathrm{Hom}_{\mathscr{C}}(Y, X)$, $g \in \mathrm{Hom}_{\mathscr{C}}(X, Y)$.

Note that one can write $X \in \mathscr{C}$ instead of $X \in Ob(\mathscr{C})$ and $f \in \mathrm{Hom}_{\mathscr{C}}(X, Y)$ can write $f : X \to Y$.

$\mathrm{Vect}_{\Bbbk}$ is the category of finite dimensional vector spaces over a field $\Bbbk$. Its objects are finite dimensional vector spaces over $\Bbbk$ and its morphisms are $\Bbbk$-linear maps. Let $V \in \mathrm{Vect}_{\Bbbk}$, call $(e_i)_{i \in I}$ a basis of V. Then each $f \in \mathrm{End}_{\mathrm{Vect}_{\Bbbk}}(V)$ has a matrix $(a_{ij})_{i,j \in I}$ in the basis $(e_i)_{i \in I}$, and the sum $\sum_{i \in I} a_{ii}$ is called the *trace* of the endomorphism f, denoted by $\mathrm{tr}(f) = \sum_{i \in I} a_{ii}$. This number does not depend on the choice of a basis of V. In particular, $\dim(V) = \mathrm{tr}(\mathrm{Id}_V)$.

Proposition 2.2. *The map* $\mathrm{tr} : \mathrm{End}_{\mathrm{Vect}_{\Bbbk}}(V) \to \Bbbk$, $f \mapsto \mathrm{tr}(f)$ *is a* $\Bbbk$*-linear, and furthermore,* $\mathrm{tr}(f \circ g) = \mathrm{tr}(g \circ f)$ *for* $f, g \in \mathrm{End}_{\mathrm{Vect}_{\Bbbk}}(V)$.

Some additional structures in the categories can be considered, it endows the categories the desired properties. We are interested in the case of a category endowed with a tensor product $\otimes_{\Bbbk}$. A *monoidal* category is a category $\mathscr{C}$ equipped with a tensor product satisfying some compatible conditions (see [6, 5]). A monoidal category $\mathscr{C}$ is $\Bbbk$-linear if for any $U, V \in \mathscr{C}$ the morphisms $\mathrm{Hom}_{\mathscr{C}}(U, V)$ form a $\Bbbk$-module and if the composition and tensor product are bilinear.

For $U, V \in \mathrm{Vect}_{\Bbbk}$, one defined a tensor product $U \otimes_{\Bbbk} V$ which is also a finite dimensional vector space and for $f \in \mathrm{Hom}_{\Bbbk}(U, V), g \in \mathrm{Hom}_{\Bbbk}(X, Y)$, the definition of $f \otimes_{\Bbbk} g \in \mathrm{Hom}_{\Bbbk}(U \otimes_{\Bbbk} X, V \otimes_{\Bbbk} Y)$ is well-defined (e.g., see [5]). Category $\mathrm{Vect}_{\Bbbk}$ equipped with the tensor product $\otimes_{\Bbbk}$ is a monoidal category. Furthermore, each $V \in \mathrm{Vect}_{\Bbbk}$ there is a vector space $V^* = \mathrm{Hom}_{\Bbbk}(V, \Bbbk) \in \mathrm{Vect}_{\Bbbk}$ called its duality and for $f \in \mathrm{Hom}(U, V)$ there is a linear map called its transpose $f^* : V^* \to U^*$ given by

$$< f^*(\alpha), u >=< \alpha, f(u) >,$$

for $\alpha \in V^*, u \in U$ where $< f^*(\alpha), u >= f^*(\alpha)(u)$. We have an isomorphism, see [5]:

Proposition 2.3. *Let* $U, V \in \mathrm{Vect}_{\Bbbk}$. *Then the map* $\lambda_{U,V} : V \otimes U^* \to \mathrm{Hom}_{\Bbbk}(U, V)$ *given by*

$$\lambda_{U,V}(v \otimes \alpha)(u) = \alpha(u)v \tag{1}$$

for $u \in U, v \in V$ *and* $\alpha \in U^*$ *is an isomorphism. In particular,* $V \otimes V^* \simeq \mathrm{End}_{\mathrm{Vect}_{\Bbbk}}(V)$.

Proof. First, suppose $\lambda_{U,V}(v \otimes \alpha) = 0$, i.e., $\alpha(u)v = 0$ for all $u \in U$. It implies that $\alpha = 0$ or $v = 0$, then $v \otimes \alpha = 0$. Next, let $f \in \mathrm{Hom}_{\Bbbk}(U, V)$ and denote $(e_i)_{i\in I}$ a basis of U, its dual basis is $(e^i)_{i\in I}$. We consider the element $\sum_{i\in I} f(e_i) \otimes e^i \in V \otimes U^*$. Then we have

$$\lambda_{U,V}\left(\sum_{i\in I} f(e_i) \otimes e^i\right)(e_j) = \sum_{i\in I} e^i(e_j)f(e_i) = f(e_j) \text{ for all } j \in I.$$

This means that $\lambda_{U,V}\left(\sum_{i\in I} f(e_i) \otimes e^i\right) = f$. Thus the map $\lambda_{U,V}$ is an isomorphism. In particular, for $U = V$ we have $V \otimes V^* \simeq \mathrm{End}_{\mathrm{Vect}_{\Bbbk}}(V)$. □

Furthermore, for each $V \in \mathrm{Vect}_{\Bbbk}$ we have an interest in the special morphisms called *evaluation map* $\overrightarrow{\mathrm{ev}}_V$ and *coevaluation map* $\overrightarrow{\mathrm{coev}}_V$ which are given by

$$\overrightarrow{\mathrm{ev}}_V\colon V^* \otimes V \to \Bbbk,\ \alpha \otimes v \mapsto <\alpha, v> \text{ and,}$$

$$\overrightarrow{\mathrm{coev}}_V\colon \Bbbk \to V \otimes V^*, 1 \mapsto \sum_i e_i \otimes e^i,$$

where $(e^i)_{i\in I}$ is the dual basis of the basis $(e_i)_{i\in I}$ of V.
Denote $\tau_{V,W} : V \otimes W \to W \otimes V, v \otimes w \mapsto w \otimes v$ the switch of V and W. Using these maps we have

Theorem 2.4. *The composition*

$$\mathrm{End}_{\mathrm{Vect}_{\Bbbk}}(V) \xrightarrow{\lambda^{-1}_{V,V}} V \otimes V^* \xrightarrow{\tau_{V,V^*}} V^* \otimes V \xrightarrow{\overrightarrow{\mathrm{ev}}_V} \Bbbk$$

defines the trace tr *on* $\mathrm{End}_{\mathrm{Vect}_{\Bbbk}}(V)$.

Proof. For $f \in \mathrm{End}_{\mathrm{Vect}_{\Bbbk}}(V)$ we can check

$$\lambda^{-1}_{V,V}(f) = (f \otimes \mathrm{Id}_{V^*}) \circ \overrightarrow{\mathrm{coev}}_V\ (1).$$

Recall that $(e_i)_{i\in I}$ is a basis of V and $(e^i)_{i\in I}$ is its dual basis. One gets

$$(f \otimes \mathrm{Id}_{V^*}) \circ \overrightarrow{\mathrm{coev}}_V\ (1) = \sum_i f(e_i) \otimes e^i,$$

then $\tau_{V,V^*}\left(\sum_i f(e_i) \otimes e^i\right) = \sum_i e^i \otimes f(e_i)$. Furthermore, $f(e_i) = \sum_j a_{ji}e_j$, hence $\overrightarrow{\mathrm{ev}}_V\left(\sum_i e^i \otimes f(e_i)\right) = \sum_{i,j} \delta_{ij}a_{ji} = \mathrm{tr}(f)$. □

It is not difficult to check $\mathrm{tr}(f \circ g) = \mathrm{tr}(g \circ f)$ for $f, g \in \mathrm{End}_{\mathrm{Vect}_{\Bbbk}}(V)$ and $\mathrm{tr}(f^*) = \mathrm{tr}(f)$.

3 Traces in pivotal categories

In previous section we have shown a way to define a trace in $\mathrm{Vect}_{\Bbbk}$ using the special morphisms $\lambda_{U,V}$, $\overrightarrow{\mathrm{ev}}_V$ and $\overrightarrow{\mathrm{coev}}_V$. In this section we explain how to define a trace in a pivotal category. We recall first the definition of a pivotal category (for details, see [2, 8]) then present the notion of a trace in the pivotal category.

3.1 Definitions

Let $\mathscr{C}$ be a monoidal category and $A, B \in \mathscr{C}$. A duality between A and B is given by a pair of morphisms $(\alpha \in \mathrm{Hom}_{\mathscr{C}}(\Bbbk, B \otimes A), \beta \in \mathrm{Hom}_{\mathscr{C}}(A \otimes B, \Bbbk))$ such that

$$(\beta \otimes \mathrm{Id}_A) \circ (\mathrm{Id}_A \otimes \alpha) = \mathrm{Id}_A \text{ and } (\mathrm{Id}_B \otimes \beta) \circ (\alpha \otimes \mathrm{Id}_B) = \mathrm{Id}_B\,.$$

A *pivotal* category (or *sovereign*) is a strict monoidal category $\mathscr{C}$, with a unity object $\Bbbk$, equipped with the data for each object $V \in \mathscr{C}$ of its *dual object* $V^* \in \mathscr{C}$ and of four morphisms

$$\overrightarrow{\mathrm{ev}}_V \colon V^* \otimes V \to \Bbbk, \qquad \overrightarrow{\mathrm{coev}}_V \colon \Bbbk \to V \otimes V^*,$$
$$\overleftarrow{\mathrm{ev}}_V \colon V \otimes V^* \to \Bbbk, \qquad \overleftarrow{\mathrm{coev}}_V \colon \Bbbk \to V^* \otimes V$$

such that $(\overrightarrow{\mathrm{ev}}_V, \overrightarrow{\mathrm{coev}}_V)$ and $(\overleftarrow{\mathrm{ev}}_V, \overleftarrow{\mathrm{coev}}_V)$ are dualities which induce the same functor duality and the same natural isomorphism $(V \otimes W)^* \cong W^* \otimes V^*$.

The family of isomorphisms

$$\Phi = \{\Phi_V = (\overleftarrow{\mathrm{ev}}_V \otimes \mathrm{Id}_{V^{**}}) \circ (\mathrm{Id}_V \otimes \overrightarrow{\mathrm{coev}}_{V^*}) : V \to V^{**}\}_{V \in \mathscr{C}}$$

is a monoidal natural isomorphism called the pivotal structure.

It is clear that $\mathrm{Vect}_{\Bbbk}$ is a pivotal category with the tensor product $\otimes_{\Bbbk}$ and the duality mentioned in previous section. Category of finite dimensional modules over a quantum group associated to a Lie (super)algebra is pivotal (see e.g., [1, 4]).

3.2 Traces in pivotal categories

Let $\mathscr{C}$ be a $\Bbbk$-linear pivotal category. For $V \in \mathscr{C}$ and $f \in \mathrm{End}_{\mathscr{C}}(V)$, one can determine a scalar $\mathrm{tr}_R(f) \in \Bbbk \simeq \mathrm{End}_{\mathscr{C}}(\Bbbk)$ by

$$\mathrm{tr}_R(f) = \overleftarrow{\mathrm{ev}}_V \circ (f \otimes \mathrm{Id}_{V^*}) \circ \overrightarrow{\mathrm{coev}}_V \in \mathrm{End}_{\mathscr{C}}(\Bbbk).$$

One calls $\mathrm{tr}_R(f)$ the *right trace* of the endomorphism f. The *left trace* of f is defined by

$$\mathrm{tr}_L(f) = \overrightarrow{\mathrm{ev}}_V \circ (\mathrm{Id}_{V^*} \otimes f) \circ \overleftarrow{\mathrm{coev}}_V \in \mathrm{End}_{\mathscr{C}}(\mathbb{k}).$$

Moreover, for $V, W \in \mathscr{C}$ and $f \in \mathrm{End}_{\mathscr{C}}(V \otimes W)$, one can also define the *partial right trace* ptr_R and the *partial left trace* ptr_L as below

$$\mathrm{ptr}_R(f) = (\mathrm{Id}_V \otimes \overleftarrow{\mathrm{ev}}_W) \circ (f \otimes \mathrm{Id}_{W^*}) \circ (\mathrm{Id}_V \otimes \overrightarrow{\mathrm{coev}}_W) \in \mathrm{End}_{\mathscr{C}}(V),$$

$$\mathrm{ptr}_L(f) = (\overrightarrow{\mathrm{ev}}_V \otimes \mathrm{Id}_W) \circ (\mathrm{Id}_{V^*} \otimes f) \circ (\overleftarrow{\mathrm{coev}}_V \otimes \mathrm{Id}_W) \in \mathrm{End}_{\mathscr{C}}(W).$$

We have the proposition

Proposition 3.1. *Let $V, W \in \mathscr{C}$. Then*

(1) $\forall f \in \mathrm{Hom}_{\mathscr{C}}(V, W)$ *and* $g \in \mathrm{Hom}_{\mathscr{C}}(W, V)$,

$$\mathrm{tr}_R(f \circ g) = \mathrm{tr}_R(g \circ f) \text{ and } \mathrm{tr}_L(f \circ g) = \mathrm{tr}_L(g \circ f),$$

(2) $\forall f \in \mathrm{End}_{\mathscr{C}}(V \otimes W)$,

$$\mathrm{tr}_R(\mathrm{ptr}_R(f)) = \mathrm{tr}_R(f) \text{ and } \mathrm{tr}_L(\mathrm{ptr}_L(f)) = \mathrm{tr}_L(f).$$

Proof. That the assumption $\mathscr{C}$ is pivotal means the left dual and the right dual in $\mathscr{C}$ coincide. This implies that

$$\overrightarrow{\mathrm{ev}} \circ (f^* \otimes \mathrm{Id}) = \overrightarrow{\mathrm{ev}} \circ (\mathrm{Id} \otimes f), \overleftarrow{\mathrm{ev}} \circ (\mathrm{Id} \otimes f^*) = \overleftarrow{\mathrm{ev}} \circ (f \otimes \mathrm{Id}) \text{ and}$$

$$(f \otimes \mathrm{Id}) \circ \overrightarrow{\mathrm{coev}} = (\mathrm{Id} \otimes f^*) \circ \overrightarrow{\mathrm{coev}}, (f^* \otimes \mathrm{Id}) \circ \overleftarrow{\mathrm{coev}} = (\mathrm{Id} \otimes f) \circ \overleftarrow{\mathrm{coev}}.$$

It follows that the first statement holds. The second one holds from the definition of the partial trace and $(V \otimes W)^* \simeq W^* \otimes V^*$. □

Remark 3.2.

1. *In the category* $\mathrm{Vect}_{\mathbb{k}}$, *the right trace and left trace coincide, i.e.,* $\mathrm{tr} = \mathrm{tr}_R = \mathrm{tr}_L$.
2. *In general, the right trace and left trace do not coincide.*

Next we represent a pivotal category constructed from the representations of a quantum group. To simplify, we consider the category of finite dimensional representations over the quantum group associated with the Lie algebra $\mathfrak{sl}(2)$.

3.3 Category of weight modules over $\mathcal{U}_q\mathfrak{sl}(2)$

Recall that a Hopf algebra means an algebra H over $\mathbb{C}$, with a unit element 1, equipped a homomorphism $\Delta : H \to H \otimes H$, an anti-homomorphism $S : H \to H$, and a homomorphism $\varepsilon : H \to \mathbb{C}$ satisfying the compatible conditions (see [5, 7]). In the terms of a Hopf algebra, Δ is called the coproduct, S the antipode and ε the counit. Quantum group $\mathcal{U}_q\mathfrak{sl}(2)$ is an associative algebra over $\mathbb{C}$ generated by the generators K, K^{-1}, E and F, subject to the following relations

$$KK^{-1} = K^{-1}K = 1, \qquad KEK^{-1} = q^2E,$$
$$KFK^{-1} = q^{-2}F, \qquad [E, F] = \frac{K - K^{-1}}{q - q^{-1}}$$

where $q \in \mathbb{C}^*$ is a complex parameter. It is a Hopf algebra with the coproduct Δ, the counity ε and the antipode S are defined by

$$\begin{aligned} &\Delta(E) = 1 \otimes E + E \otimes K, && \varepsilon(E) = 0, && S(E) = -EK^{-1}, \\ &\Delta(F) = K^{-1} \otimes F + F \otimes 1, && \varepsilon(F) = 0, && S(F) = -KF, \\ &\Delta(K) = K \otimes K, && \varepsilon(K) = 1, && S(K) = K^{-1}. \end{aligned}$$

We consider the finite dimensional modules over $\mathcal{U}_q\mathfrak{sl}(2)$ in which each module splits as a direct sum of highest weight modules which will be defined below. A such module is called a *weight module*. For a module V and a scalar $\lambda \neq 0$, denote by V^λ the subspace of all vectors $v \in V : Kv = \lambda v$. The scalar λ is called a *weight* of V if $V^\lambda \neq \{0\}$. One has $EV^\lambda \subset V^{q^2\lambda}$ and $FV^\lambda \subset V^{q^{-2}\lambda}$. We recall the definition of a highest weight module in [5]. Let V be a $\mathcal{U}_q\mathfrak{sl}(2)$-module and λ be a scalar. A non zero element $v \in V$ is a *highest weight vector* of weight λ if $Ev = 0$ and if $Kv = \lambda v$. V is a *highest weight module* of highest weight λ if it is generated by a highest weight vector of weight λ.

It is proven that any non zero finite dimensional $\mathcal{U}_q\mathfrak{sl}(2)$-module contains a highest weight vector (see [5]).

Suppose V, W are the highest weight modules over $\mathcal{U}_q\mathfrak{sl}(2)$. Using the maps coproduct Δ and antipode S one can determine the action of $\mathcal{U}_q\mathfrak{sl}(2)$ on $V \otimes W$ and $V^* = \mathrm{Hom}_{\mathbb{C}}(V, \mathbb{C})$: for $v \in V, w \in W$ and $a \in \mathcal{U}_q\mathfrak{sl}(2)$,

$$a.(v \otimes w) = \sum a_{(1)}v \otimes a_{(2)}w,$$

where we used the notation $\Delta(a) = \sum a_{(1)} \otimes a_{(2)}$, and for $\alpha \in V^*, a \in \mathcal{U}_q\mathfrak{sl}(2)$,

$$a.\alpha = \alpha \circ S(a),$$

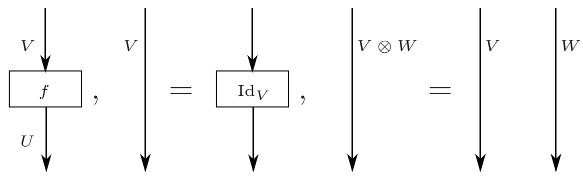

Figure 1 – Representation of the morphisms $f, \mathrm{Id}_V, \mathrm{Id}_{V\otimes W}$

(for more details, see [5]). With these actions $V \otimes W$ and V^* are the modules over $\mathcal{U}_q\mathfrak{sl}(2)$. Let $\mathscr{C}$ be the category of finite dimensional weight modules over $\mathcal{U}_q\mathfrak{sl}(2)$. It is known that $\mathscr{C}$ is a pivotal category (see [8]), its pivotal structure is given by

$$\overrightarrow{\mathrm{ev}}_V\colon e^i \otimes e_j \mapsto e^i(e_j) = \delta_{ij}, \qquad \overrightarrow{\mathrm{coev}}_V\colon 1 \mapsto \sum_{i=1}^{n} e_i \otimes e^i,$$

$$\overleftarrow{\mathrm{ev}}_V\colon e_j \otimes e^i \mapsto e^i(Ke_j), \qquad \overleftarrow{\mathrm{coev}}_V\colon 1 \mapsto \sum_{i=1}^{n} e^i \otimes K^{-1}e_i$$

where $(e_i)_{i=1..n}$ is a basis of V and $(e^i)_{i=1..n}$ is the dual basis of $(e_i)_{i=1..n}$. Note that the pivotal structure is determined by element $K \in \mathcal{U}_q\mathfrak{sl}(2)$. We can check that $\mathrm{tr}_R(\mathrm{Id}_V) = \mathrm{tr}_L(\mathrm{Id}_V) = \mathrm{tr}(\mathrm{Id}_V) = n = \dim V$.

4 Application

In this section we represent how to apply the trace in a pivotal category to construct an invariant of a link diagram (for more details, see [9]). We describe the method through an example.

First, we recall a technique of presenting morphisms of a tensor category by planar diagrams (see [5]). Let $\mathscr{C}$ be a tensor category. We present a morphism $f : U \to V$ by a box with two vertical arrows oriented downwards as the first component in Figure 1. The tensor product and the composition of f and g are represented in Figure 2. For the identity of V^*, we represent by the vertical arrow directed upwards colored by V. Figure 4 represents the braiding and its inverse of the category $\mathscr{C}$. It is known that these graphs form a tensor category, and there exists a functor F (called Reshetikhin-Turaev functor) from the category of the graphs to the category $\mathscr{C}$ (see [9]).

Then let Γ be a planar diagram of a link L. We decompose Γ into the elementary graphs mentioned above.

Finally, by applying the functor F one gets the morphism $F(\Gamma) \in \mathrm{End}_{\mathscr{C}}(\mathbb{C})$ as in the decomposition of the graph Γ the elementary graphs at the bottom

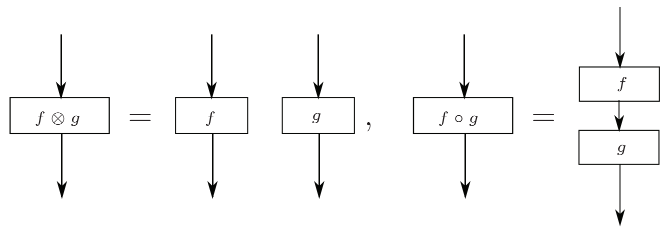

Figure 2 – Representation of the tensor product and the composition

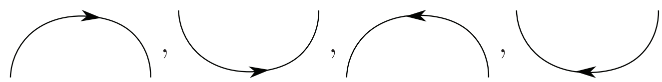

Figure 3 – Morphisms $\overrightarrow{\mathrm{ev}}_V, \overrightarrow{\mathrm{coev}}_V, \overleftarrow{\mathrm{ev}}_V, \overleftarrow{\mathrm{coev}}_V$

are the cups and these at the tops are the caps. Moreover, $\mathrm{End}_{\mathscr{C}}(\mathbb{C}) \simeq \mathbb{C}$. Thus, for each diagram Γ we determine a complex number $F(\Gamma)$, this number depends only on the isotopy class of the link L.

An example of the calculus for the Hopf link Γ is illustrated in Figure 5 where

$$F(\Gamma) = (\overrightarrow{\mathrm{ev}}_V \otimes \overleftarrow{\mathrm{ev}}_W) \circ (\mathrm{Id}_{V^*} \otimes c_{W,V} \otimes \mathrm{Id}_{W^*}) \circ (\mathrm{Id}_{V^*} \otimes c_{V,W} \otimes \mathrm{Id}_{W^*}) \circ (\overleftarrow{\mathrm{coev}}_V \otimes \overrightarrow{\mathrm{coev}}_W).$$

It is clear that $F(\Gamma) \in \mathrm{End}_{\mathscr{C}}(\mathbb{C}) \simeq \mathbb{C}$.

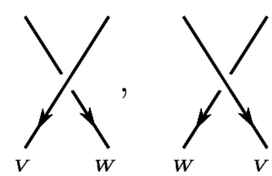

Figure 4 – Morphisms $c_{V,W}, c_{V,W}^{-1}$

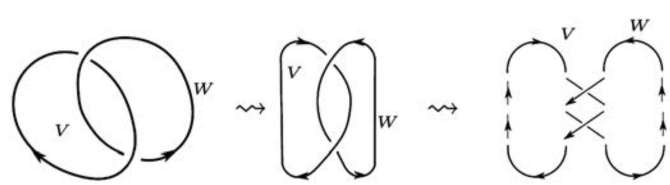

Figure 5 – The decomposition of the Hopf link

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SỰ TỔNG QUÁT HÓA CỦA VẾT TRONG CÁC PHẠM TRÙ THEN CHỐT

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TÓM TẮT

Ta đã biết vết là một tính chất đặc trưng của các tự đồng cấu của một không gian véc tơ và nó là một bất biến của các ma trận biểu diễn các tự đồng cấu đó. Khái niệm vết đã được tổng quát hóa vào lý thuyết phạm trù và các tính chất của nó được sử dụng để xây dựng một số bất biến tô pô lượng tử. Trong bài báo này chúng tôi chỉ ra sự tổng quát của khái niệm vết trong các phạm trù và vài ứng dụng của khái niệm này.

Từ khóa: *Vết, đại số Hopf ruy băng, bất biến của các dãy.*